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Entanglement and relative phase created by decay

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Abstract

We discuss an exactly solvable model for the creation of entanglement between two subsystems by the observation of decay products. The system consists of two identical decaying boson modes, and the decay channels are observed through a beam splitter. For a reasonable class of initial states the decay process is completely decoupled from the buildup of the relative phase. Exact expressions are derived for the distribution over the two output channels, and for the conditional density matrix after a given detection history.

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1. Introduction

Entanglement is one of the several fundamental features of quantum mechanics, with no classical counterpart. Two quantum systems in a pure state are entangled when the state of the combined system cannot be factorized into product states pertaining to the separate systems. The possibility of entanglement arises naturally from the superposition principle. In the case of two spatially separated subsystems, entanglement is known to lead to measured correlations between the subsystems that cannot be understood in terms of local properties of each subsystem alone. One may be tempted to believe that for two systems to be in an entangled state it is necessary that they share a common past. This is true for two particles arising from a common source, or separating after a collision. Practical examples are two photons created by parametric downconversion [1], or polarized atoms exiting their interaction region. A slightly more involved possibility is that each system of an entangled pair has been interacting with a common partner system. More fanciful possibilities have recently been realized, where two entangled pairs of photons separate in such a way that two photons, one out of each pair, are brought to interfere, leading to entanglement of the remaining two photons [2]. In this paper we analyse the creation of entanglement between two decaying systems by the detection of their decay products in interference. The two systems are modelled as boson modes, and can be thought of as two single-mode radiation cavities or two Bose–Einstein condensates. Particles emitted from the systems enter the two input ports of a beam splitter and detectors are attached to the output ports. Finite detection efficiencies are allowed for. In the case of Bose–Einstein condensates, this is a situation where spontaneous symmetry breaking gives rise to a relative phase between the condensates [3]. We consider a class of initial states where the detection statistics and the resulting buildup of the relative phase during a detection history can be solved analytically. The method is based on quantum trajectories, generalized to include imperfect detection efficiency.

2. Single histories of decaying system

2.1. Perfect detection efficiency

The evolution of an open quantum system can often be described by a quantum master equation for its density matrix $\hat{\rho}$. The simplest form of this equation is [4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} \equiv \mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{1}{2}\Gamma(\hat{C}^{\dagger}\hat{C}\hat{\rho} + \hat{\rho}\hat{C}^{\dagger}\hat{C}) + \Gamma\hat{C}\hat{\rho}\hat{C}^{\dagger} \tag{1}$$

where the operator $\hat{C}$ represents the effect of a quantum jump and Γ is a measure of the jump rate. The last term in equation (1) describes the gain in the final state after quantum jumps. In the prototype case of spontaneous emission, the operator $\hat{C}$ is the lowering operator of the atom, which transforms the excited state to the lower state. For a decaying mode of the radiation field in a cavity, $\hat{C} = \hat{a}$ represents the annihilation of a photon from the mode. Quantum master equations are valid when the correlations of the outside world decay so rapidly that within the decay time the state of the system does not change appreciably [5].

An initially pure state of the system cannot be expected to remain pure. Loss of information to the outside world increases the entropy of the state. However, the density matrix $\hat{\rho}$ can always be represented as an ensemble of time-dependent pure states, so that the density matrix follows after ensemble averaging. This is the basis of the method of quantum trajectories [6–8]. Each realization of the pure state of the system is specified by the precise specification of the instants of time at which the quantum jumps occurred. The method is derived by separating the evolution operator $\mathcal{L}$ in equation (1) as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{2}$$

with $\mathcal{L}_1\hat{\rho} = \Gamma\hat{C}\hat{\rho}\hat{C}^{\dagger}$ the gain term. Then equation (1) can be expressed as an integral equation

$$\hat{\rho}(T) = \mathrm{e}^{\mathcal{L}_0 T}\hat{\rho}(0) + \int_0^T \mathrm{d}t\, \mathrm{e}^{\mathcal{L}_0(T-t)}\mathcal{L}_1\hat{\rho}(t) \tag{3}$$

which after iteration leads to a formal solution of the master equation in the form of an expansion in $\mathcal{L}_1$. The integrand in each term can be viewed as a possible pure-state history of the system with given instants of the jumps [9]. The first term $\hat{\tilde{\rho}}_0(T) \equiv \mathrm{e}^{\mathcal{L}_0 T}\hat{\rho}(0)$ in (3) represents that no jump occurred in the interval $[0, T]$; the second term describes all histories with the last jump at time t. The strength of the first term $p_0(T) = \mathrm{Tr}\,\hat{\tilde{\rho}}_0(T)$ has the physical significance of the probability that no jump occurred during the interval $[0, T]$. The pure-state single histories are convenient for numerical simulation of single runs of the evolution of open quantum systems.

The integral equation (3) can also be used to derive the conditional density matrix for a given history of the jumps. The situation that precisely N jumps have occurred at the successive time instants $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N$, within the count interval $[0, T]$, is described by the integrand of the Nth iteration. The corresponding contribution to the total density matrix at time T is then

$$\hat{\tilde{\rho}}_N(t_1, t_2, \ldots, t_N; T) = \mathrm{e}^{\mathcal{L}_0(T-t_N)}\mathcal{L}_1\mathrm{e}^{\mathcal{L}_0(t_N-t_{N-1})}\ldots\mathcal{L}_1\mathrm{e}^{\mathcal{L}_0 t_1}\hat{\rho}(0). \tag{4}$$

The corresponding normalized density matrix describes the conditional state of the system, given this detection history. It can be written as the normalized version $\hat{\tilde{\rho}}_N/w_N$ of (4), with

$$w_N(t_1, t_2, \ldots, t_N; T) = \mathrm{Tr}\, \hat{\tilde{\rho}}_N(t_1, t_2, \ldots, t_N; T). \tag{5}$$

The strength (5) represents the probability density for precisely N jumps in the count interval $[0, T]$ at the indicated time instants. The probability $p_N(T)$ for N jumps in the count interval is obtained after integration of w_N over the ordered time instants $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N$.

2.2. *Imperfect detection efficiency*

For our purposes we also need the statistics of detected jumps [10] in the case of imperfect detection efficiency. The corresponding single histories of the evolution of the system can no longer be expressed as pure-state evolution [11]. For simplicity we assume that the jumps are detected with the uniform efficiency η $(0 \leqslant \eta \leqslant 1)$. We separate the evolution operator $\mathcal{L}$ as in equation (2), but we redefine the detection part as $\mathcal{L}_1 = \eta\Gamma\hat{C}\hat{\rho}\hat{C}^\dagger$. The complementary part of the gain term is included in $\mathcal{L}_0$. With this redefinition of the partial evolution operators the integral equation (3) remains valid. However, the single histories now represent a specific number of *detected* jumps. Undetected jumps still contribute to $\mathcal{L}_0$ to an amount $1 - \eta$, with the result that single histories are no longer represented by pure states. In this case, equations (4) and (5) give the density matrix and the probability density for a given number of detected jumps at the indicated time instants. In the limiting case of full detection efficiency $\eta = 1$, the standard pure-state trajectories are recovered. In the opposite limiting case of $\eta = 0$, there is no difference between the solution of the full master equation (1) and the density matrix $\hat{\tilde{\rho}}_0(t)$ during a detection-free period, since $\mathcal{L} = \mathcal{L}_0$.

3. Decay of single boson mode

3.1. *Arbitrary initial state*

The topic of this paper is the interfering decay of two boson modes. In order to fix the notation, we first discuss the quantum trajectories with limited detection efficiency in the case of a single boson mode. Particles are leaking to the outside world, where they have a chance η to be detected. The system can represent a mode of the radiation field or a Bose–Einstein condensate. A single decaying system is described by the master equation [12]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} \equiv \mathcal{L}\hat{\rho} = -i\omega[\hat{a}^\dagger\hat{a}, \hat{\rho}] - \frac{1}{2}\Gamma(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \Gamma\hat{a}\hat{\rho}\hat{a}^\dagger. \tag{6}$$

This is identical to equation (1), with the Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$, and with the jump operator $\hat{C} = \hat{a}$ replaced by the annihilation operator of a particle from the mode. We are interested in the conditional density matrix corresponding to a specific detection history. The initial density matrix $\hat{\rho}(0)$ can be expanded in coherent states $|\alpha\rangle$ in terms of the Glauber–Sudarshan P-function as [5]

$$\hat{\rho}(0) = \int \mathrm{d}_2\alpha\, |\alpha\rangle\langle\alpha| P(\alpha) \tag{7}$$

where the integration extends over the complex α-plane. In order to find P we introduce the characteristic function

$$\chi(\lambda) = \langle \mathrm{e}^{\lambda\hat{a}^\dagger}\mathrm{e}^{-\lambda^*\hat{a}}\rangle = \mathrm{Tr}\, \hat{\rho}(0)\, \mathrm{e}^{\lambda\hat{a}^\dagger}\mathrm{e}^{-\lambda^*\hat{a}} \tag{8}$$

so that knowledge of χ as a function of λ determines the density matrix $\hat{\rho}$. Then the distribution function P defined as the two-dimensional Fourier transform of χ,

$$P(\alpha) = \frac{1}{\pi^2} \int d_2\lambda\, \chi(\lambda)\, e^{\lambda^*\alpha - \lambda\alpha^*} \tag{9}$$

determines $\hat{\rho}(0)$ as in (7). The function P is real and normalized. On the other hand, P is not necessarily positive definite, so one cannot interpret it as a probability distribution function. In fact, when $\hat{\rho}(0)$ is a number state, the function χ is a polynomial, so that P contains derivatives of a delta function. The possibly singular behaviour of P presents no problems in the present paper.

The representation (7) allows us to express the evolution of the total density matrix and the state during a detection-free interval in terms of the evolution of coherent states. It is easy to check that the total evolution of a coherent state described by the master equation (6) is determined by the identity

$$e^{\mathcal{L}T} |\alpha\rangle\langle\alpha| = |\alpha(T)\rangle\langle\alpha(T)| \tag{10}$$

with the time-dependent coherent state given by $\alpha(T) = \alpha \exp[-(i\omega + \Gamma/2)T]$. For a detection efficiency η, the effect of a detection is described by the operator $\mathcal{L}_1\hat{\rho} = \eta\hat{a}\hat{\rho}\hat{a}^\dagger$. This also defines the evolution operator for a detection-free time period as $\mathcal{L}_0 \equiv \mathcal{L} - \mathcal{L}_1$. Since obviously $\mathcal{L}_1|\alpha\rangle\langle\alpha| = \eta|\alpha|^2|\alpha\rangle\langle\alpha|$, the detection-free evolution of an initial coherent state is given by

$$e^{\mathcal{L}_0 T} |\alpha\rangle\langle\alpha| = \exp(-\eta|\alpha|^2(1 - e^{-\Gamma T}))|\alpha(T)\rangle\langle\alpha(T)|. \tag{11}$$

Hence, the initial condition (7) gives the time-dependent solution

$$\hat{\rho}(T) = \int d_2\alpha\, |\alpha(T)\rangle\langle\alpha(T)| P(\alpha) \tag{12}$$

whereas the contribution to the density matrix corresponding to a detection-free interval is

$$\hat{\hat{\rho}}_0(T) = \int d_2\alpha\, |\alpha(T)\rangle\langle\alpha(T)| P(\alpha)\, \exp(-\eta|\alpha|^2(1 - e^{-\Gamma T})). \tag{13}$$

The trace of (13) gives the zero-detection probability

$$p_0(T) = \int d_2\alpha\, P(\alpha)\, \exp(-\eta|\alpha|^2(1 - e^{-\Gamma T})). \tag{14}$$

Since the coherent state $|\alpha\rangle$ is eigenstate of the annihilation operator with eigenvalue α, the effect of the detection of a particle as described by $\mathcal{L}_1$ can simply be accounted for. The contribution (4) to the density matrix corresponding to N detections in the interval $[0, T]$ at the instants $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N$ is now easily evaluated, with the result

$$\hat{\hat{\rho}}_N(t_1, t_2, \ldots, t_N; T) = \int d_2\alpha\, |\alpha(T)\rangle\langle\alpha(T)| \times P(\alpha)\, \exp(-\eta|\alpha|^2(1 - e^{-\Gamma T})) \prod_{i=1}^{N} (\Gamma\eta|\alpha|^2 e^{-\Gamma t_i}). \tag{15}$$

The trace of (15) gives the N-fold probability density $w_N(t_1, t_2, \ldots, t_N; T)$ for precisely N detections at the given instants of time, and the ratio $\hat{\hat{\rho}}_N/w_N$ gives the conditional density matrix, given that N particles have been detected. After an N-fold integration of w_N over the ordered time instants, one obtains the normalized probability distribution for precisely N detected particles in the interval $[0, T]$, with the result

$$p_N(T) = \int d_2\alpha\, P(\alpha) \frac{1}{N!} (\eta|\alpha|^2(1 - e^{-\Gamma T}))^N\, \exp(-\eta|\alpha|^2(1 - e^{-\Gamma T})). \tag{16}$$

The distribution (16) has the form of the average of a Poisson distribution, with the P function as the effective probability distribution over the coherent-state index α. The mean value is

$$\bar{N} = \int \mathrm{d}_2\alpha\, P(\alpha)\eta|\alpha|^2(1 - \mathrm{e}^{-\Gamma T}) \tag{17}$$

which is determined by the time integral of the detection rate $\eta\Gamma|\alpha(t)|^2$. The probability distribution p_N can be sub-Poissonian, since the function $P(\alpha)$ is not necessarily positive definite. A sub-Poissonian distribution occurs when the difference between the variance and the mean value of N is negative. This difference is found in the form

$$\Delta N^2 - \bar{N} = \eta^2(1 - \mathrm{e}^{-\Gamma T})^2\left(\int \mathrm{d}_2\alpha\, P(\alpha)|\alpha|^4 - \left(\int \mathrm{d}_2\alpha\, P(\alpha)|\alpha|^2\right)^2\right) \tag{18}$$

which is negative when also the initial distribution of the number of particles in the mode is sub-Poissonian.

3.2. Poisson distribution of particle numbers

The expressions simplify considerably when the function $P(\alpha)$ is non-zero only for a single value of $|\alpha|$. Then the initial state is specified by the distribution over the phase of $\alpha = r\,\exp(-\mathrm{i}\phi_A)$. In equations (12)–(16) we can make the replacement $\int \mathrm{d}_2\alpha\, P(\alpha) \to \int \mathrm{d}\phi_A\, g_A(\phi_A)$, with g_A a normalized distribution over the phase ϕ_A. The time-dependent probability distribution over the number of particles in the mode is Poissonian, with mean value $\langle n\rangle = r^2 \exp(-\Gamma T)$. In this case, the N-fold distribution function for detections at the instants $t_1, t_2, \ldots, t_N$ takes the simple form

$$w_N(t_1, t_2, \ldots, t_N; T) = \exp(-\eta r^2(1 - \mathrm{e}^{-\Gamma T}))\prod_{i=1}^{N}(\Gamma\eta r^2\mathrm{e}^{-\Gamma t_i}). \tag{19}$$

Since the detection probability is determined only by the value of r, this distribution is independent of the phase distribution. The corresponding contribution to the density matrix (15) takes the factorized form

$$\hat{\tilde{\rho}}_N(t_1, t_2, \ldots, t_N; T) = w_N(t_1, t_2, \ldots, t_N)\int \mathrm{d}\phi_A\, g_A(\phi_A)|\alpha(T)\rangle\langle\alpha(T)| \tag{20}$$

where now the time-dependent value of the coherent-state parameter is given by $\alpha(T) = r\exp[-\mathrm{i}\phi_A - (\mathrm{i}\omega + \Gamma/2)T]$. The integral in (20) is the conditional density matrix after N detections. Notice that this conditional density matrix is identical to the unconditioned density matrix (12) in this special case. This is due to the fact that the detection probability is independent of the phase ϕ_A. This also implies that subsequent particle detections are uncorrelated, so that the distribution (16) reduces to a Poisson distribution. The mean value of the number of detections is $\bar{N} = \eta r^2(1 - \mathrm{e}^{-\Gamma T})$. The simplest example is an initial coherent state $|r\,\exp(-\mathrm{i}\phi_0)\rangle$, so that g_A is effectively a delta function. Then the state remains a pure state at all times, given by the coherent state $|r\,\exp(-\mathrm{i}\phi_0 - \mathrm{i}\omega T - \Gamma T/2)\rangle$.

When the system is a Bose–Einstein condensate, states with different particle numbers do not superpose, and the density matrix must be diagonal in the particle number. The initial state is diagonal in the number of particles only when the phase ϕ_A is uniformly distributed. Hence in equations (12)–(16) we can make the replacement $\int \mathrm{d}_2\alpha\, P(\alpha) \to \int \mathrm{d}\phi_A/2\pi$. The time-dependent state (12) is then a Poissonian statistical mixture of number states.

4. Decay of two boson modes

4.1. Two representations

Now we consider two independently decaying boson modes A and B. The combined density matrix obeys the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}\omega[\hat{a}^\dagger\hat{a},\hat{\rho}] - \frac{1}{2}\Gamma(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \Gamma\hat{a}\hat{\rho}\hat{a}^\dagger \\ - \mathrm{i}\omega[\hat{b}^\dagger\hat{b},\hat{\rho}_2] - \frac{1}{2}\Gamma(\hat{b}^\dagger\hat{b}\hat{\rho} + \hat{\rho}\hat{b}^\dagger\hat{b}) + \Gamma\hat{b}\hat{\rho}\hat{b}^\dagger. \tag{21}$$

When the density matrix $\hat{\rho}(0)$ at time 0 can be expressed as a product $\hat{\rho}_A\hat{\rho}_B$ of terms corresponding to the separate systems, also the combined Glauber–Sudarshan function $P(\alpha,\beta)$ can be written as a product $P_A(\alpha)P_B(\beta)$. Then the state of the modes is unentangled. Since the two modes evolve independently, the time-dependent solution of the master equation (21) is just the product of two solutions of the form (12). Hence, the two modes remain uncorrelated at all times. Also when emitted particles are detected, while the products from both modes are distinguishable, no entanglement can arise.

The situation is quite different, however, when the emitted bosons from the two modes are observed in interference. Since it is undetermined whether the particle arose from mode A or from mode B, entanglement can arise. The effect of a detection on the density matrix can be described by the substitution

$$\hat{\rho}(t) \to \hat{c}(\phi)\hat{\rho}(t)\hat{c}(\phi)^\dagger \tag{22}$$

in terms of the detection operator

$$\hat{c}(\phi) = \tfrac{1}{\sqrt{2}}(\hat{a} + \mathrm{e}^{\mathrm{i}\phi}\hat{b}) \tag{23}$$

that is the superposition of an annihilation of a particle from either of the two modes. After detection of a particle as described by an operator of the form (23), the conditional density matrix no longer factorizes. This is the basic mechanism for the creation of the relative phase between two Bose–Einstein condensates when they are observed in interference [3, 13]. Spatial interference between two condensates has been observed in a number of experiments [14, 15].

We assume that particles emitted from both sources are combined in a beam splitter and detected at the two output ports. The setup is sketched in figure 1. Particle detection is described by the two orthogonal detection operators

$$\hat{c}_\pm = (\hat{a} \pm \hat{b})/\sqrt{2}. \tag{24}$$

This same detection scheme has been analysed by Castin and Dalibard [3] in the case of two condensates. The operators $\hat{c}_\pm$ obey the standard commutation rules $[\hat{c}_\pm,\hat{c}_\pm^\dagger] = 1$, $[\hat{c}_\pm,\hat{c}_\mp] = [\hat{c}_\pm,\hat{c}_\mp^\dagger] = 0$. The master equation (21) can be rewritten in terms of the operators $\hat{c}_\pm$ rather than in terms of $\hat{a}$ and $\hat{b}$, with the result

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}\omega[\hat{c}_+^\dagger\hat{c}_+,\hat{\rho}] - \frac{1}{2}\Gamma(\hat{c}_+^\dagger\hat{c}_+\hat{\rho} + \hat{\rho}\hat{c}_+^\dagger\hat{c}_+) + \Gamma\hat{c}_+\hat{\rho}\hat{c}_+^\dagger \\ - \mathrm{i}\omega[\hat{c}_-^\dagger\hat{c}_-,\hat{\rho}_2] - \frac{1}{2}\Gamma(\hat{c}_-^\dagger\hat{c}_-\hat{\rho} + \hat{\rho}\hat{c}_-^\dagger\hat{c}_-) + \Gamma\hat{c}_-\hat{\rho}\hat{c}_-^\dagger \equiv \mathcal{L}\hat{\rho}. \tag{25}$$

Since this is just a different form of the same equation, the solutions of (25) are identical to those of (21). No entanglement can be created by the full evolution given by the right-hand side of (25).

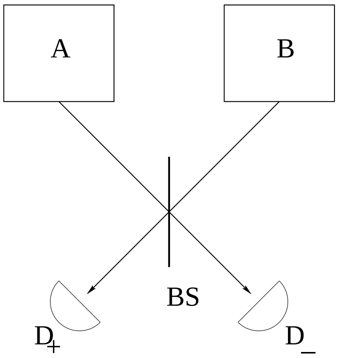

Figure 1. Sketch of setup of two decaying modes. The outputs are mixed by a beam splitter BS and particle detectors D_+ and D_- are attached to the output ports.

4.2. Detection-induced correlation

The result of a detection history can be described as a direct generalization of the one-mode case discussed in section 3. A detection history is now described by specifying for each detection within a time interval $[0, T]$ the time instant and the detection channel. Just as in equation (2), we separate the evolution operator $\mathcal{L}$ in equation (25), with $\mathcal{L}_1\hat{\rho} = \Gamma(\eta_+\hat{c}_+^\dagger\hat{\rho}\hat{c}_+ + \eta_-\hat{c}_-^\dagger\hat{\rho}\hat{c}_-) \equiv \mathcal{L}_{1+} + \mathcal{L}_{1-}$, with η_1 and η_2 the two detection efficiencies. These two terms describe two types of quantum jumps corresponding to the two detection channels. We consider a factorized initial state in terms of the combined P-function as

$$\hat{\rho}(0) = \int \mathrm{d}_2\alpha\, \mathrm{d}_2\beta\, |\alpha, \beta\rangle\langle\alpha, \beta| P_A(\alpha) P_B(\beta) \tag{26}$$

where $|\alpha, \beta\rangle$ denote two-mode coherent states. These states are eigenfunctions of $\hat{a}$ and $\hat{b}$, and therefore also of $\hat{c}_\pm$. The eigenvalue relations are $\hat{c}_\pm|\alpha, \beta\rangle = \gamma_\pm|\alpha, \beta\rangle$, with $\gamma_\pm = (\alpha \pm \beta)/\sqrt{2}$. Therefore, the variables $\gamma_\pm$ in equations (27) and (28) are functions of α and β. Now it is easy to evaluate the contribution to the time-dependent density matrix corresponding to zero detections in the interval $[0, T]$, and the result is

$$\begin{aligned}\hat{\tilde{\rho}}_0(T) \equiv \mathrm{e}^{\mathcal{L}_0 T}\hat{\rho}(0) = \int \mathrm{d}_2\alpha\, \mathrm{d}_2\beta\, |\alpha(T), \beta(T)\rangle\langle\alpha(T), \beta(T)| \\ \times P_A(\alpha) P_B(\beta)\, \exp(-(\eta_+|\gamma_+|^2 + \eta_-|\gamma_-|^2)(1 - \mathrm{e}^{-\Gamma T})).\end{aligned} \tag{27}$$

The time dependence of the coherent states $|\alpha(T), \beta(T)\rangle$ is the same as defined in section 3. The probability for zero detections is the trace of (27), so that

$$p_0(T) = \int \mathrm{d}_2\alpha\, \mathrm{d}_2\beta\, P_A(\alpha) P_B(\beta)\, \exp(-(\eta_+|\gamma_+|^2 + \eta_-|\gamma_-|^2)(1 - \mathrm{e}^{-\Gamma T})). \tag{28}$$

When the two efficiencies $\eta_\pm$ are unequal and the distributions P_A and P_B have a finite width, the last line in (27) does not factorize in the variables α and β. This indicates that entanglement has been created by the simple fact that a detection experiment has been performed with a null outcome. This is particularly obvious when a detector is attached only to the output port D_+. Then after an interval with zero detection, the entangled state is described by the ratio of equation (27) and (28), with the substitution $\eta_- = 0$. This situation is reminiscent of a measurement scheme discussed by Plenio *et al* [16]. Here an entangled subradiant state of two atoms is created by a null result of a photon measurement leaking out of a cavity.

A detection history is specified as $t_1, s_1; t_2, s_2; \ldots; t_N, s_N$, with t_i the time instant and $s_i = D_\pm$ the channel of the ith detection. Each detection tends to enhance the entanglement between the two modes, even though they share no common past. This will be illustrated in an exactly solvable special case in section 5.

5. Initial states with Poisson particle statistics

In analogy to what we did in section 3.2 for one boson mode, we now assume that the initial state of each subsystem is characterized by P-functions that are non-zero only for the single value $|\alpha| = |\beta| = r$, while the efficiencies $\eta_+ = \eta_- = \eta$ are equal. Hence, the initial state is fully symmetric. Setting $\alpha = r\,\exp(-\mathrm{i}\phi_A)$, $\beta = r\,\exp(-\mathrm{i}\phi_B)$, one notices that the integrations effectively extend only over the phases weighed by a factorized two-phase distribution $g_A(\phi_A)g_B(\phi_B)$. The solution of the master equation (21) (or (25)) is then

$$\hat{\rho}(T) = \int \mathrm{d}\phi_A\,\mathrm{d}\phi_B\,|\alpha(T),\beta(T)\rangle\langle\alpha(T),\beta(T)|g_A(\phi_A)g_B(\phi_B). \tag{29}$$

Formally, by iterating equation (3), this density matrix can be unraveled into a summation and integration over all possible detection histories. A detection at the output ports $D_\pm$ gives rise to a multiplicative factor proportional to $|\gamma_+|^2 = 2r^2\cos^2(\phi/2)$, or $|\gamma_-|^2 = 2r^2\sin^2(\phi/2)$, with $\phi = \phi_A - \phi_B$ the relative phase between the two modes. For the contribution to the density matrix corresponding to a given detection history we find

$$\begin{aligned}\hat{\tilde{\rho}}_N(t_1,s_1;t_2,s_2;\ldots;t_N,s_N;T) &= \exp(-2\eta r^2(1-\mathrm{e}^{-\Gamma T}))\\ &\times\prod_{i=1}^{n}(2\Gamma\eta r^2\mathrm{e}^{-\Gamma t_i})\int \mathrm{d}\phi_A\,\mathrm{d}\phi_B\,|\alpha(T),\beta(T)\rangle\langle\alpha(T),\beta(T)|\\ &\times g_A(\phi_A)g_B(\phi_B)\,\cos^{2n}(\phi/2)\,\sin^{2m}(\phi/2).\end{aligned} \tag{30}$$

Here n and m are the number of detections in each channel, so that $n + m = N$ is the total number. This effect reflects the change of the state at a quantum measurement. The trace of (30) gives the distribution function for the indicated detection history, with the factorized expression

$$\begin{aligned}&w_N(t_1,s_1;t_2,s_2;\ldots;t_N,s_N;T)\\ &\quad= F(n,m)\,\exp(-2\eta r^2(1-\mathrm{e}^{-\Gamma T}))\prod_{i=1}^{n}(2\Gamma\eta r^2\mathrm{e}^{-\Gamma t_i}).\end{aligned} \tag{31}$$

This is the probability distribution for precisely N detections in the interval $[0, T]$ at the indicated instants of time t_i and the indicated detection channels $s_i = D_\pm$. The information on the distribution over the two detection channels is contained in the coefficients $F(n,m)$. The remaining r-dependent factor represents the probability distribution for precisely N detections, summed over both detection channels. This expression is quite similar to equation (19) for the N-fold distribution function of a single decaying mode, the only difference being that now the detection rate is doubled compared to the single-mode case. The statistics of the total number of detections is given by a Poisson distribution, with mean value $2\eta r^2(1-\mathrm{e}^{-\Gamma T})$. The coefficients $F(n,m)$ are defined as

$$F(n,m) = \int \mathrm{d}\phi\,g(\phi)\,\cos^{2n}(\phi/2)\,\sin^{2m}(\phi/2) \tag{32}$$

with $g(\phi) = \int \mathrm{d}\phi_B\,g_A(\phi_B+\phi)g_B(\phi_B)$ the normalized distribution function over the relative phase ϕ in the initial state. The coefficient $F(n,m)$ represents the probability that the N detections occur in an ordered sequence $s_1, s_2, \ldots, s_N$ over the two detection channels $D_\pm$, with precisely n detections in the channel D_+ and $m = N - n$ detections in the channel D_-.

Obviously, the detection statistics depends only on the initial distribution of the relative phase ϕ, not on the absolute phases of the two modes, nor on their population. More importantly, this probability depends only on the total number of detections (n and m) in each channel, not on their time order. This makes it easy to obtain the probability distribution for the possible partitions (n, m) over the two detection channels, with $m = N - n$. Since for given n and m the total number of time orderings of the N detections is a binomial coefficient, we find the expression for the probability that n out of the N detections occurred in the channel D_+,

$$p(n, m) = \binom{N}{n} F(n, m). \tag{33}$$

For a given value of N, these probabilities indeed add up to one. Naturally, the specific values of the coefficients $F(n, m)$ and the probabilities $p(n, m)$ depend on the initial distribution $g(\phi)$ of the relative phase. In order to evaluate these quantities, it is convenient to introduce the branching ratios $f_\pm(n, m)$, defined as the probabilities that after a detection history with n and m particles in the two output channels, the next detected particle is found in the channel $D_\pm$. From the significance of the coefficients $F(n, m)$ it is obvious that

$$f_+(n, m) = \frac{F(n + 1, m)}{F(n, m)} \qquad f_-(n, m) = \frac{F(n, m + 1)}{F(n, m)}. \tag{34}$$

These ratios obey the sum rule $f_+(n, m) + f_-(n, m) = 1$. The values of these branching ratios depend exclusively on the distribution function $g(\phi)$ over the relative phase that specifies the initial state. Once the branching ratios have been calculated, the definition (34) shows that the coefficients $F(n, m)$ can be written as a product of branching ratios corresponding to the sequence of detections of particles. The fact that the resulting product does not depend on the specific order of the detections restricts the possible values of the branching ratios. For instance, they must obey the identity $f_+(n, m) f_-(n + 1, m) = f_-(n, m) f_+(n, m + 1) = F(n + 1, m + 1)/F(n, m)$. In fact, a moment's reflection reveals that knowledge of the ratios $f_+(n, 0)$ for all values of n is sufficient to evaluate all branching ratios, and thereby all the coefficients $F(n, m)$. Mathematically, this is equivalent to saying that these branching ratios are sufficient for reproducing the relative phase distribution $g(\phi)$.

The normalized conditional density matrix, given a detection history (n_0, m_0) in both output channels, is given by

$$\hat{\rho}(n_0, m_0; T) = \int \mathrm{d}\phi_A \, \mathrm{d}\phi_B \, |\alpha(T), \beta(T)\rangle\langle\alpha(T), \beta(T)| \\ \times g_A(\phi_A) g_B(\phi_B) \cos^{2n_0}(\phi/2) \sin^{2m_0}(\phi/2)/F(n_0, m_0). \tag{35}$$

It is equal to the ratio of (30) and (31), it is fully specified by the detection numbers n_0 and m_0 and it does not depend on the instants or the ordering of the detections. Equation (35) shows as the effect of each detection that the combined P distribution function is multiplied by $\cos^2(\phi/2)$ or $\sin^2(\phi/2)$, apart from normalization.

Knowledge of the coefficients $F(n, m)$ also allows us to derive expressions for the conditional statistics of the number of detections, following a given initial number of n_0 and m_0 detections in the two channels. Since $F(n, m)$ represents the probability for a specific order of the detections, the conditional probability that the next N detections follow a specific sequence $s_1, s_2, \ldots, s_N$ is given by

$$F(n, m \mid n_0, m_0) = F(n_0 + n, m_0 + m) \mid F(n_0, m_0) \tag{36}$$

which again depends only on the partition (n, m) of the conditional sequence. For the conditional probability distribution over the partitions we thus find

$$p(n, m \mid n_0, m_0) = \binom{N}{n} F(n_0 + n, m_0 + m)/F(n_0, m_0). \tag{37}$$

These same expressions follow from the conditional density matrix (35) as the initial state.

6. Special cases

6.1. Initial coherent states

A trivial special case of section (5) arises when both modes are in a coherent state. Then the phase distribution function $g(\phi) = \delta(\phi - \phi_0)$ with ϕ_0 the difference of the phases of α and β. This is a natural situation when the particles are photons leaking out of cavities. However, for Bose condensates, this initial state is not possible, since it is not diagonal in the total number of particles. In this case, the branching ratios (34) are $f_+ = \cos^2(\phi_0/2)$ and $f_- = \sin^2(\phi_0/2)$ for all values of n and m. The distribution (33) over the two channels is binomial, with average numbers of detections $N\,\cos^2(\phi/2)$ and $N\,\sin^2(\phi/2)$ in both channels. When the phase difference is $\phi_0 = \pi/2$, the branching ratios are equal to $1/2$ and the probability distribution (33) over the channels is

$$p(n, m) = \frac{1}{2^N}\binom{N}{n} \tag{38}$$

just as the conditional distribution (37). Likewise, the conditional density matrix (35) after (n, m) detections is equal to the unconditioned density matrix (29), so that the density matrix is not affected by the detection. The measurement process is highly classical, and no entanglement is created. The average value is $\overline{n} = N/2$ and the standard deviation is equal to $\Delta n = \sqrt{N}/2$. In the limit of large detection numbers N, the distribution (38) is well approximated by a Gaussian, specified by these numbers. Hence, the relative width decreases with N. The conditional density matrix given a certain detection history is identical to the unconditioned solution of the master equation (21), which represents a product of two coherent states at all times.

6.2. Initial uniform phase distribution

The opposite extreme case occurs when the initial phase distributions of both modes are completely uniform. Then initially both modes are diagonal in the number states, with a Poissonian distribution. The unconditional state that solves the master equation is given by (29), with $g_A = g_B = 1/2\pi$. This state is unentangled and diagonal in the number state of both modes at all times. This uniform-phase state has been considered for the analysis of the buildup of a fixed relative phase between two Bose condensates in a continuous measurement [13]. In fact, the detection statistics depends only on the distribution function of the relative phase, which is also uniform, so that $g(\phi) = 1/2\pi$. Initially, there is no preference for either channel, and the two branching ratios $f_+(0, 0) = f_-(0, 0) = 1/2$ are equal. However, after a number of detections, a preference is created for the channel that had already the most detections. This effect has been discussed for Bose condensates starting from initial number states in the limit of large numbers [3]. Here we derive exact expressions for the statistical distribution, while accounting for the limited detection efficiency, and the decay of the modes during the detection interval.

From the definition (34) of f_+, it is easy to show for the case of a uniform phase distribution that $f_+(n, 0) = (2n + 1)/(2n + 2)$. Furthermore, we have argued that the product of branching ratios along a detection sequence depends only on the initial and final values of (n, m), not on the specific sequence. Together with the sum rule $f_+ + f_- = 1$, this allows us to obtain simple exact expressions for the branching ratios

$$f_+(n,m) = \frac{2n+1}{2(N+1)} \qquad f_-(n,m) = \frac{2m+1}{2(N+1)} \tag{39}$$

with $N = n + m$. This yields exact expressions for the coefficients F in the form

$$F(n,m) = \frac{(2n)!(2m)!}{2^{2N} N!n!m!}. \tag{40}$$

This leads in turn to the exact result for the probability distribution (33) of N detected particles over the two channels,

$$p(n,m) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2m}{m}. \tag{41}$$

Whereas the binomial distribution (38) has maximal values for $n = m = N/2$, the distribution (41) is maximal for $n = 0$ or $m = 0$, so that the detected particles tend to bunch in one channel. Hence the detection histories bifurcate into sequences with practically all detections in the same output channel. For these most probable detection histories, with most particles in one channel, the phase distribution becomes very narrow. The conditional density matrix after a detection sequence (n, m) with $n \ll m$ (or $n \gg m$) is given by equation (35), which contains a phase-dependent function that strongly peaks at $\phi = 0$ (or at $\phi = \pi$). This state is highly entangled, provided that the remaining number of particles per mode $r^2 \exp(-\Gamma T)$ is still appreciable. This illustrates the strong correlation between successive detections.

For large particle numbers, the distribution (41) can be represented by a simple continuous approximation. From the Gaussian limit of the binomial distribution one finds that the binomial coefficients occurring in (41) can be approximated by

$$\binom{2n}{n} \approx 2^{2n} \frac{1}{\sqrt{\pi n}}. \tag{42}$$

In the limit of large N values, the distribution (41) is therefore well represented by the continuous normalized distribution

$$q(\nu) = 1/\pi\sqrt{\nu(1-\nu)} \tag{43}$$

over the range $0 \leqslant \nu \leqslant 1$ of the variable $\nu = n/N$. The variable ν and the complementary variable $\mu = m/N = 1-\nu$ determine the number of detections in the two output channels. The average value and the standard deviation of ν can be directly evaluated after the substitution $\nu = \cos^2(\phi/2)$, with the result

$$\bar{\nu} = \frac{1}{2} \qquad \Delta\nu = \frac{1}{2\sqrt{2}} \tag{44}$$

which confirms that the standard deviation of n is of the order of the total number of particles N.

This limiting distribution (43) can also be understood geometrically by considering the situation that two fields with amplitudes proportional to $a = \exp(-i\phi_A)/2$ and $b = \exp(-i\phi_B)/2$ enter the input port of the beam splitter. The amplitudes of the two output channels are then proportional to the sum and difference of these fields, which corresponds to intensities proportional to $\nu = \cos^2(\phi/2)$ and $\mu = \sin^2(\phi/2)$. A uniform distribution over the relative phase $\phi = \phi_A - \phi_B$ reproduces the distribution $q(\nu)$. This is illustrated in figure 2.

While during the detection history, the distribution over the relative phase ϕ becomes very narrow, the phases ϕ_A and ϕ_B of the two modes remain completely undetermined. Conversely, the state remains diagonal in the total particle number in the two modes, whereas the difference in particle number in both modes becomes undetermined. This can be seen explicitly from

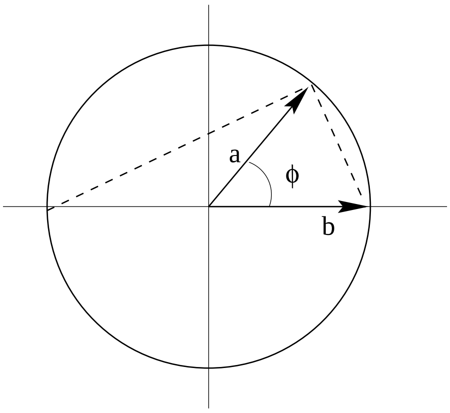

Figure 2. Geometrical picture of distribution of relative intensity ν. The lengths of the dotted lines determine the intensities $\nu = |a + b|^2$ and $\mu = |a - b|^2$. A uniform distribution over the relative phase ϕ reproduces the distribution (43) over ν.

the expression (35), in the special case that g_A and g_B are uniform, and $n_0 \gg m_0$, so that the distribution for the relative phase ϕ strongly peaks at the value 0.

7. Conclusions and discussion

We have obtained exact expressions for the conditional density matrix of two decaying boson systems when the decay products are brought into interference by a beam splitter. The expressions become particularly simple when the initial states are diagonal in the number states, with a Poisson distribution. In this case, the total decay summed over both output channels is an autonomous process which is not affected by the quantum measurement. On the other hand, the distribution over the relative phase is completely determined by the specific detection history and the statistics of the detected particles in both channels can be completely solved. This is illustrated in equation (34) for the branching ratios for the next detection and in equation (35) for the conditional density matrix, following a detection history with n particles in the channel D_+ and m particles in the channel D_-. The description is equally valid for photons leaking from cavities, or for bosonic atoms leaking from two condensates. In either case, the model illustrates the spontaneous buildup of the relative phase during the decay process. For bosonic atoms, no off-diagonality in the total number of particles can exist for initially isolated systems, and then it cannot arise during the decay process either. This means that the phase of each subsystem separately must remain completely undetermined, even when the relative phase attains a specific value. This implies that the states of the systems get entangled, even though they have never been in contact directly.

For an initially uniform distribution of the relative phase, the probability distribution of the detected particles over the two channels can be solved exactly, and the result is given in equation (41). Since this probability distribution is normalized, we obtain as a byproduct the sum rule for binomial coefficients,

$$\sum_{n=0}^{N} \binom{2n}{n} \binom{2(N-n)}{N-n} = 2^{2N}. \tag{45}$$

This distribution expresses strong correlations between successive detections. An accidental asymmetry between the two channels tends to be amplified as the detection history proceeds, leading to a well-defined relative phase. It is remarkable that no operator for the relative phase enters the description [17]. In fact, the detection process is described by the operators $c_{\pm}$, which are special cases of the phase-dependent superposition operator (23). It has been pointed out that states with maximum visibility in interference experiments based on such field operators are the eigenstates of the effective number operator $c^{\dagger}(\phi)c(\phi)$ [18]. It is therefore significant that after a detection history the conditional density matrix becomes diagonal in the eigenstates of $c_{+}^{\dagger}c_{+}$ and $c_{-}^{\dagger}c_{-}$. A detection history can be considered as a quantum measurement of these particle numbers, which forces the state into the corresponding eigenstates.

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